Critical analysis of derivative dispersion relations at high energies

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Abstract

We discuss some formal and fundamental aspects related with the replacement of integral dispersion relations by derivative forms, and their practical uses in high energy elastic hadron scattering, in particular pp and $\bar{p}p$ scattering. Starting with integral relations with one subtraction and considering parametrizations for the total cross sections belonging to the class of entire functions in the logarithm of the energy, a series of results is deduced and our main conclusions are the following: (1) except for the subtraction constant, the derivative forms do not depend on any additional free parameter; (2) the only approximation in going from integral to derivative relations (at high energies) concerns to assume as zero the lower limit in the integral form; (3) the previous approximation and the subtraction constant affect the fit results at both low and high energies and therefore, the subtraction constant can not be disregarded; (4) from a practical point of view, for single-pole Pomeron and secondary reggeons parametrizations and center-of-mass energies above 5 GeV, the derivative relations with the subtraction constant as a free fit parameter are completely equivalent to the integral forms with finite (non-zero) lower limit. A detailed review on the conditions of validity and assumptions related with the replacement of integral by derivative relations is also presented and discussed.

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Contents

1 Introduction

Elastic hadron-hadron scattering, the simplest soft diffractive process, constitutes one of the hardest challenges in high-energy physics. Despite all the success of QCD concerning hard and also semi-hard processes, the large distances involved in the elastic sector (and soft processes in general) demand a nonperturbative approach and, presently, we do not know how to calculate an elastic scattering amplitude in a purely nonperturbative QCD context. At this stage, phenomenology is very important, but must be based on the constraints imposed on the scattering amplitudes by some rigorous theorems, deduced from general principles of the underlying local quantum field theory, namely, Lorentz Invariance, Unitarity, Analyticity and Crossing.

Dispersion relations play an important role in several areas of Physics, both as a practical tool and a formal theoretical result. In special, for particle-particle and particle-antiparticle interactions, they are consequences of the principles of Analyticity and Crossing [1]. In this context, they correlate real

and imaginary parts of crossing even (+) and odd (−) amplitudes, which in turn are expressed in terms of the scattering amplitudes for a given process and its crossed channel, for example, $a + b$ and $a + \bar{b}$:

$$F_{ab} = F_{+} + F_{-} \qquad F_{a\bar{b}} = F_{+} - F_{-}. \tag{1}$$

Among the physical quantities that characterize high-energy elastic hadron-hadron scattering, the total cross section (optical theorem) and the ρ parameter (related with the phase of the amplitude) are just expressed in terms of forward real and imaginary parts of the amplitude [2],

$$\sigma_{\mathrm{tot}}(s) = \frac{\mathrm{Im}\, F(s, t = 0)}{2k\sqrt{s}}, \tag{2}$$

$$\rho(s) = \frac{\mathrm{Re}\, F(s, t = 0)}{\mathrm{Im}\, F(s, t = 0)}, \tag{3}$$

where s and k are the center-of-mass energy squared and the momentum, respectively, and t is the four-momentum transfer squared. Therefore, a natural and well founded framework for investigating the behaviors of σ_{tot} and ρ, as function of the energy, is by means of dispersion relations. On one hand, from analytical parametrizations for σ_{tot} and fits to the corresponding experimental data, the ρ parameter may be determined (analytically or numerically) and compared to the experimental data and/or extrapolated to higher energies. On the other hand, σ_{tot} and ρ may be analytically connected through the dispersion relations and, therefore, both may be determined from global fits to the experimental data (that is, simultaneous fits to σ_{tot} and ρ data), improving the statistical ensemble in terms of degree of freedom (F).

For particles and antiparticles, the pp and $\bar{p}p$ scattering correspond to the highest energy interval with available data, reaching $\sqrt{s} \sim 63$ GeV for pp (CERN ISR) and $\sqrt{s} \sim 2$ TeV for $\bar{p}p$ (Fermilab Tevatron). From experiments that are being conducted at the Brokhaven RHIC it is expected data on pp scattering at $\sqrt{s}$: 50 - 500 GeV, and in the near future the CERN LHC will provide data on pp scattering at 16 TeV. It is highly expected that, in short term, these novel experiments will allow crucial tests for several phenomenological models and, among them, the analytical models, based on the use of dispersion relation techniques, certainly play a central role.

The use of dispersion relations in the investigation of scattering amplitudes may be traced back to the end of fifties, when they were introduced in the form of *integral* relations. Despite the important results that have been obtained since then, one limitation of the integral forms is their non-local character: in order to obtain the real part of the amplitude, the imaginary part must be known for all values of the energy. Moreover, the class of functions that allows

analytical integration is limited. By the end of the sixties and beginning of the seventies, there appeared the dispersion relations in a *differential* form, and they provided new insights in the dispersion relation techniques. However, as we shall discuss, there still remain some questions related with both the formal replacement of integral by the derivative forms and their practical uses in high energies, as, for example, the effects of the approximations considered, the correct expression of derivative form (depending on the class of functions involved) and the role of the subtraction constant. We understand that these aspects must be made clear before any practical use of the derivative relations.

In this work we critically discuss the situation concerning the derivative relations and present some answers for the above mentioned questions. The manuscript is organized as follows. In Section 2 we briefly review some historical facts and results about the integral and derivative relations, stressing the problems we are interested in. In Section 3 we first demonstrate that, for the class of functions that are entire in the logarithm of the energy, except for the subtraction constant, the derivative relations do not depend on any additional free parameter. We then review the main results in the literature, obtained through different approaches, in connection with the corresponding assumptions and classes of functions involved. In Section 4, making use of a pomeron-reggeon parametrization, we analyze the practical effects of the approximations involved in the replacement of the integral by the derivative forms. In particular, we show that, for this class of functions, the derivative form with the subtraction constant as a free fit parameter is equivalent to the integral form for $\sqrt{s}$ above 5 GeV. The conclusions and some critical remarks are the contents of Section 5.

2 Historical summary and main points

In this Section we briefly review some results related with the replacement of *integral dispersion relations* (IDR) by *derivative dispersion relations* (DDR), stressing the points that we shall discuss in the next sections. We are interested in the high-energy region, specifically $\sqrt{s} > 5$ GeV and, as commented in our introduction, the practical tests will be performed with the pp and $\bar{p}p$ experimental data on σ_{tot} and ρ. For these scatterings, the experimental data above $\sqrt{s} \sim 20$ GeV indicate a slow increase of σ_{tot}, roughly as $\sim \ln^2 s$ and that $\sigma_{\mathrm{tot}}^{\bar{p}p} - \sigma_{\mathrm{tot}}^{pp} \sim 0$ as $s \to \infty$. These conditions allow the use of integral dispersion relations with only one subtraction, and in the standard form they are usually expressed in terms of the energy E and momentum p of the incoming proton in the laboratory system [2,3]. Since we shall be interested in the region $\sqrt{s} > 5$ GeV, we can approximate $s = 2m(E + m) \sim 2mE$ with an error $< 5\%$, which decreases as the energy increases. In this case the standard IDR, with

poles removed, are given by

$$\mathrm{Re}\, F_+(s) = K + \frac{2s^2}{\pi} P \int_{s_0}^{+\infty} \mathrm{d}s' \frac{1}{s'(s'^2 - s^2)} \,\mathrm{Im}\, F_+(s'), \tag{4}$$

$$\mathrm{Re}\, F_-(s) = \frac{2s}{\pi} P \int_{s_0}^{+\infty} \mathrm{d}s' \frac{1}{(s'^2 - s^2)} \,\mathrm{Im}\, F_-(s'), \tag{5}$$

where K is the subtraction constant, and for pp and $\bar{p}p$ scattering, $s_0 = 2m^2 \sim 1.8$ GeV^2. It should be noted that for the even amplitude the relations with one and two subtractions are equal and for the odd case the relations without subtraction and with one subtraction are equal.

Through Eqs. (1) to (5) the physical quantities σ_{tot} and ρ may be simultaneously investigated. By means of IDR, important results have been obtained since the beginning of the seventies, as for example in the works by Bourrely and Fischer [4], Amaldi et al. [5], Block and Cahn [2], Kluit and Timmermans [6], Augier et al. (UA 4/2 Collaboration) [7], Kang, Valin and White [8], Bertini et al. [9] and many others. However, as commented in our introduction, the inconveniences with the IDR concern their non-local character and the limited number of functions that allows analytical integration, and therefore, error propagation from the fit parameters. On the other hand, under some conditions, the above integral forms may be replaced by quasi-local ones, expressed in a derivative form and called derivative dispersion (or analyticity) relations. The first result in that direction appeared indicated in the works by Gribov and Migdal [10], and afterwards, the DDR were treated in more detail by Bronzan [11], Jackson [12], Bronzan, Kane and Sukhatme [13]. In particular, the forms deduced by the last authors are given by

$$\mathrm{Re}\, F_+(s) = s^\alpha \tan\left[\frac{\pi}{2}\left(\alpha - 1 + \frac{\mathrm{d}}{\mathrm{d}\ln s}\right)\right] \frac{\mathrm{Im}\, F_+(s)}{s^\alpha}, \tag{6}$$

$$\mathrm{Re}\, F_-(s) = s^\alpha \tan\left[\frac{\pi}{2}\left(\alpha + \frac{\mathrm{d}}{\mathrm{d}\ln s}\right)\right] \frac{\mathrm{Im}\, F_-(s)}{s^\alpha}, \tag{7}$$

where α is a real parameter. Soon after, based on the Sommerfeld-Watson-Regge representation and other assumptions, Kang and Nicolescu introduced the following expression for the derivative relations [14]:

$$\frac{\mathrm{Re}\, F_+(s)}{s} = \left[\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s} + \frac{1}{3}\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right)^3\right.$$

$$
+\frac{2}{5}\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right)^5+\cdots\Big]\frac{\mathrm{Im}\,F_+(s)}{s}, \tag{8}
$$

$$
\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\frac{\mathrm{Re}\,F_-(s)}{s}=-\Big[1-\frac{1}{3}\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right)^2
-\frac{1}{45}\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right)^4-\cdots\Big]\frac{\mathrm{Im}\,F_-(s)}{s}, \tag{9}
$$

which, according to the authors, reduces to the Bronzan, Kane and Sukhatme result for a decreasing power form of $\mathrm{Im}[F_-(s)/s]$ [14]. In the last years, these equations have been extensively used in the detailed analysis on analytical models by Cudell et al. [15] and by the COMPETE Collaboration [16].

The DDR, in the form introduced by *Bronzan, Kane and Sukhatme* (BKS), have been criticized by several authors [17,18,19,20] and analyzed and discussed in detail by Fischer, Kolář [21] and Vrkoč [22]. Despite all these works, we understand that two aspects related with the replacement of the IDR, Eqs. (4) and (5), by the DDR, Eqs. (6), (7), or (8), (9) and with their practical uses are yet unclear in the literature.

One aspect concerns the origin, role and range of the exponent α in the BKS relations, which does not appear in the formulas by Kang and Nicolescu. For example, in the paper by BKS this parameter is allowed to have any real value [13], but in References [18,23] (see also [21]) it must lie in the interval $0 < \alpha < 2$. According to some authors, "the choice of α different from 1 has no practical advantage" [21] and other authors have used it as a free fit parameter [13,24], which demands some physical interpretation. The calculation by Eichmann and Dronkers, in 1974, lead to DDR that do not depend on α [17], but this parameter appears in a recent review by Kolář and Fischer [25]. To our knowledge, with the exception of the above quoted considerations of α as a free fit parameter, all the other practical uses of the BKS relations are made by assuming $\alpha = 1$. However, $\alpha = 1$ leads to a singularity in the BKS expansion of the odd amplitude, since it contains the term $\sec^2(\pi\alpha/2)$ [13]. As we shall discuss in detail in Section 3.2, some of these results and statements are not contradictories, because they refer either to different energy regimes (finite or asymptotic s) or to different classes of functions, for example, entire functions in the logarithm of the energy, or functions satisfying general principles from axiomatic field theory, etc... Moreover, some results have been obtained in different historical contexts in terms of the highest energies reached in experiments, leading to different mathematical assumptions on the asymptotic conditions.

Another aspect concerns the subtraction constant in the singly subtracted IDR, Eq. (4), which is the starting formula in the work by Bronzan, Kane and Sukhatme. That constant does not appear in all the above derivative forms and, to our knowledge, neither in almost all their practical uses, including the detailed works by the COMPETE collaboration. Yet, to our knowledge, the only exception concerns the recent works [26,27,28], where we have performed analysis of the experimental data available, including cosmic-ray estimations for the total cross sections, by using DDR with the subtraction constant as a free fit parameter and have shown that it affects the fit results at both low and high energies; therefore, the subtraction constant can not be disregarded. We have also shown that, for functions that are entire in the logarithm of the energy (Taylor expansion), the derivative forms do not depend on the parameter α [28]. More recently, Cudell, Martynov and Selyugin have introduced a new representation for the DDR, intended for low energies as well, and they also refer to the possible role of the subtraction constant in the derivative form [29].

Based on the above considerations, in the next sections we investigate some formal and practical aspects related with the replacement of the IDR by the DDR. Specifically we shall demonstrate that, for entire functions in the logarithm of the energy, the derivative forms do not depend on the α parameter and shall discuss both the approximations involved and the important practical and formal role of the subtraction constant. Some of these results have already been presented elsewhere [28].

3 Derivative dispersion relations

In this section we first show that the subtraction constant is preserved when the IDR are replaced by the DDR and that, for functions entire in the logarithm of the energy, the derivative forms do not depend on any additional free parameter. Next, based on the formulas displayed, we present a detailed discussion on the conditions of validity of the different results mentioned in the previous section, in connection with the assumptions involved.

3.1 Formal Derivation

Let us consider the even amplitude, Eq. (4). By defining $s' = \mathrm{e}^{\xi'}$, $s = \mathrm{e}^{\xi}$ and $g(\xi') = \mathrm{Im}\, F_+(\mathrm{e}^{\xi'})/\,\mathrm{e}^{\xi'}$, we express

$$\mathrm{Re}\, F_+(\mathrm{e}^{\xi}) - K = \frac{2\,\mathrm{e}^{2\xi}}{\pi} P\int_{\ln s_0}^{+\infty} \frac{g(\xi')\,\mathrm{e}^{\xi'}}{\mathrm{e}^{2\xi'} - \mathrm{e}^{2\xi}}\,\mathrm{d}\xi' = \frac{\mathrm{e}^{\xi}}{\pi} P\int_{\ln s_0}^{+\infty} \frac{g(\xi')}{\sinh(\xi' - \xi)}\,\mathrm{d}\xi'.$$

Assuming that g is an *analytical function of its argument*, we perform the expansion

$$g(\xi') = \sum_{n=0}^{\infty} \frac{\mathrm{d}^n}{\mathrm{d}\xi'^n} g(\xi')\bigg|_{\xi'=\xi} \frac{(\xi'-\xi)^n}{n!}$$

and then, the integration term by term in the above formula. At the high energy limit, we consider the *essential approximation* $s_0 = 2m^2 \to 0$, so that $\ln s_0 \to -\infty$. With these conditions, we obtain

$$\mathrm{Re}\, F_+(\mathrm{e}^\xi) - K = \frac{\mathrm{e}^\xi}{\pi} \sum_{n=0}^{\infty} \frac{g^{(n)}(\xi)}{n!} P\int_{-\infty}^{+\infty} \frac{(\xi'-\xi)^n}{\sinh(\xi'-\xi)}\,\mathrm{d}\xi'.$$

Now, defining $y = \xi' - \xi$, the above formula may be put in the form

$$\mathrm{Re}\, F_+(\mathrm{e}^\xi) - K = \mathrm{e}^\xi \sum_{n=0}^{\infty} \frac{g^{(n)}(\xi)}{n!} I_n,$$

where,

$$I_n = \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{y^n}{\sinh y}\,\mathrm{d}y.$$

For n even, $I_n = 0$ and for n odd, we consider the integral

$$J(a) = \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{ay}}{\sinh y}\,\mathrm{d}y = \tan\left(\frac{a\pi}{2}\right),$$

so that,

$$I_n = \frac{\mathrm{d}^n}{\mathrm{d}a^n} J(a)\bigg|_{a=0} = \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{ay}\, y^n}{\sinh y}\,\mathrm{d}y\bigg|_{a=0}.$$

With this we have

$$\begin{aligned}\mathrm{Re}\, F_+(\mathrm{e}^\xi) - K &= \mathrm{e}^\xi \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}a^n} \tan\left(\frac{\pi a}{2}\right)\bigg|_{a=0} \frac{\mathrm{d}^n}{\mathrm{d}\xi'^n} g(\xi')\bigg|_{\xi'=\xi} \\ &= \mathrm{e}^\xi \tan\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\xi}\right) g(\xi)\end{aligned}$$

and, therefore,

$$\frac{\operatorname{Re} F_+(s)}{s} = \frac{K}{s} + \tan\left[\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right]\frac{\operatorname{Im} F_+(s)}{s}, \tag{10}$$

where the series expansion is implicit in the tangent operator. With analogous procedure for the odd relation we obtain

$$\operatorname{Re} F_-(s) = \tan\left[\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right]\operatorname{Im} F_-(s),$$

or

$$\frac{\operatorname{Re} F_-(s)}{s} = \tan\left[\frac{\pi}{2}\left(1 + \frac{\mathrm{d}}{\mathrm{d}\ln s}\right)\right]\frac{\operatorname{Im} F_-(s)}{s}. \tag{11}$$

We see that, with the exception of the subtraction constant, these formulas do not depend on any additional free parameter, or, they correspond to a "particular" case of the BKS relations for $\alpha = 1$. We stress that the same result is obtained if, following BKS, the derivation begins with an integration by parts without free parameter, as can be easily verified. We shall return to this point in Section 3.2.

In practical uses the trigonometric operators are expressed by the corresponding series. With the substitution in the odd case

$$\tan\left[\frac{\pi}{2}\left(1 + \frac{\mathrm{d}}{\mathrm{d}\ln s}\right)\right] \quad \to \quad -\cot\left[\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right],$$

and, by expanding the series *around the origin*, we obtain

$$\frac{\operatorname{Re} F_+(s)}{s} = \frac{K}{s} + \left[\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s} + \frac{1}{3}\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right)^3 + \frac{2}{5}\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right)^5 + \cdots\right]\frac{\operatorname{Im} F_+(s)}{s}, \tag{12}$$

$$\frac{\operatorname{Re} F_-(s)}{s} = -\frac{2}{\pi}\int\left\{\left[1 - \frac{1}{3}\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right)^2 - \frac{1}{45}\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}\ln s}\right)^4 - \cdots\right]\frac{\operatorname{Im} F_-(s)}{s}\right\}\mathrm{d}\ln s, \tag{13}$$

which is formally equivalent to the results introduced by Kang and Nicolescu, Eqs. (8) and (9), except for the presence of the subtraction constant. These expansions have been used in the Ref. [26], that includes power and logarithmic dependences in the parametrizations.

We have just demonstrated three formal and important results: (1) the subtraction constant is preserved when the IDR are replaced by DDR and, therefore, in principle, can not be disregarded; (2) except for the subtraction constant, the DDR with entire functions in the logarithm of the energy, do not depend on any additional free parameter; (3) the only approximation involved in the replacement concerns the lower limit in the IDR, namely, $s_0 = 2m^2 \to 0$, which represents a high-energy approximation. We shall return to these points in Section 4, when treating practical uses of the DDR.

3.2 Discussion

With focus on the DDR as expressed by Eqs. (6) to (13), let us discuss here the different results mentioned in Section 2, in connection with the corresponding conditions of validity and assumptions involved. We shall roughly follow a chronological approach (see also [25]).

The first derivative form appeared in the beginning of 1968 in the context of the Regge Theory. By investigating the cuts associated with the Pomeron in the high energy limit ($s \to \infty$) and at low momentum transfers, Gribov and Migdal introduced a derivative representation for the Watson-Sommerfeld integral, which, mathematically, corresponds to the first term in the expansion (12) (without the subtraction constant), namely

$$\frac{\operatorname{Re} F_+(s,t)}{s} = \left[\frac{\pi}{2}\frac{\partial}{\partial \ln s}\right]\frac{\operatorname{Im} F_+(s,t)}{s}.$$

In the beginning of the seventies, Bronzan [11], followed by Jackson [12], introduced the first form involving the tangent operator, Eqs. (10) and (11), independent of the parameter α. The arguments by Jackson were based on the formal representation of the Taylor series by an exponential operator,

$$f(z+\lambda) = \exp\left\{\lambda\frac{d}{dz}\right\}f(z),$$

an on the high energy limit, in order to avoid branch points and cuts (see [12] for details). Therefore, the basic assumptions concern entire functions and the high energies.

The first direct connection between IDR and DDR appeared in 1974 in the work by BKS [13]. Starting from the singly subtracted IDR, Eq. (4) (neglecting the subtraction constant), the authors mention an integration by parts, which leads to an expression containing the parameter α. That may be obtained by multiplying and dividing Eq. (4) by s^α and then integrating by parts. In addition to the high-energy approximation, represented by the lower limit $s_0 \to 0$, a Taylor expansion of $\mathrm{Im}\, F_+/s^\alpha$ is also assumed, so that the series may be integrated term by term (uniform convergence), leading to Eqs. (6) and (7). We should note that the expansion considered by BKS in [13] is around the point $\pi(\alpha - 1)/2$ for the even amplitude and $\pi\alpha/2$ in the odd case, which leads to the previously mentioned singularity for $\alpha = 1$. Differently, Eqs. (8), (9) and (12), (13) and all the other cases treated here and discussed in what follows, refer to expansions around the origin.

In the same year, Kang and Nicolescu introduced the series expansion (8) and (9), which, according to the authors, can be derived "from the analyticity relation given by the Sommerfeld-Watson-Regge representation" and are asymptotic relations [14]. The point here was to treat the possibility that the amplitude could include not only simple poles in the complex angular momentum plane but, in particular, logarithm dependences in the odd amplitude, representing the Odderon [30]. The association of this concept with a complicated singularity at J =1, demanded a specific contour for the Watson-Sommerfeld-Regge transformation, which could be translated by the need of subtraction in the dispersion relation [14]. That, in turn, could be represented by the derivative factor, $(\pi/2)d/dlns$, in the odd case, Eq. (9) [31]. As we have shown, the DDR by Kang-Nicolescu and that by BKS have the same mathematical structure, once the cotangent operator (Section 3.1) could be well defined for the specific class of functions involved. We shall return to this important point in Section 5.

At that time, the derivative approach was strongly criticized by several authors [17,18,19,20], in the sense that it had no practical interest, for instance, because "the mathematical condition for the convergence of the series excludes all cases of physical interest" [17]. However, it should be noted that some of these criticisms were connected with the physical situation at that time, some directed to attempts to extend the method to low energies (for example, as in [32]) and/or to some possible excessive optimism about the derivative approach [33]. As effective contributions from all these criticisms we may mention the following results, which refer always to the even amplitude.

Eichmann and Dronkers treated the case corresponding to $\alpha = 1$, showing that Eq. (10) is valid only for some class of entire functions of $\ln s$ [17]. Specifically, with our notation of Sec. 3.1, they considered the series expansion of

$$\tan\left[\frac{\pi}{2}\frac{d}{d\xi}\right]g(\xi)$$

and have proved that, if this series converges, then $g(\xi)$ must be an entire function of ξ and must satisfy $|g(\xi)| \leq C\exp\{|\xi|\}$, C a constant. Conversely, if $g(\xi)$ is an entire function of ξ and satisfies $|g(\xi)| \leq C\exp\{(1-\epsilon)|\xi|\}$, $\epsilon > 0$, than the tangent series converges and may be represented by the integral relation (4), with $s_0 = 0$. Basically, the proof consisted in expanding $g(\xi)$ in Taylor series and compare the result with that obtained by performing the same expansion of $g(\xi)$ in the integral (4). They have also shown that the method does not apply in the resonance region.

Heidrich and Kazes considered the parameter α and showed that the convergence of the series demands that α must lie between 0 and 2 [18]. As before, it is assumed that $\mathrm{Im}\, F_+(s)/s^\alpha$ may be expanded in Taylor series and that, in turn, integrated term by term. That leads to the evaluation of an integral depending on $y = \xi - \xi'$, in the form

$$\frac{1}{\pi}\ln\coth\frac{1}{2}|y|\ \ e^{(\alpha-1)y}y^n|_{-\infty}^{+\infty} + \frac{1}{\pi}\int_{-\infty}^{+\infty} dy\frac{e^{(\alpha-1)y}}{\sinh y}y^n,$$

so that the first term is finite only for $0 < \alpha < 2$. They have also shown that Eq. (6) is violated in the energy intervals between two branch points on the cut. It may be noted that a criterion of convergence based on the convergence radius of the Taylor expansion was later shown to be incorrect by Fischer and Kolář [21].

From 1976 to 1987, Fischer and Kolář, in a series of seminal papers [21], developed a rigorous study of the derivative approach in connection with general principles and theorems from axiomatic quantum field theory [34]. The authors introduced classes of functions satisfying, among other properties, polynomial bound and the Froissart-Martin bound, and corresponding to a wider class of functions than that represented by entire functions in the logarithm of the energy. The analysis was focused on the region of asymptotic energies and the main formal result was that the derivative relations are valid if the tangent series is replaced by its first term. Although that demands the existence of the high-energy limits of certain physical quantities [21], the class of functions involved includes the majority of functions of interest in physical applications. For our purposes, we recall some results obtained by the authors about the role and range of the parameter α. For $x = \ln s$, let $F(x)$ be a function belonging to the above mentioned class of functions [21]. For $\alpha = 1$, the entire function $\mathcal{F}(z)$ which extends the function $F(x)$ to the complex z plane, obeys the bound

$$|\mathcal{F}(z)| \leq \epsilon e^{|z|} + C(\epsilon),$$

for every $\epsilon > 0$, where $C(\epsilon)$ is a constant that depends on ϵ. On the other hand, for $\alpha \neq 1$, if the series

$$\tan\left[\frac{\pi}{2}(\alpha - 1 + \frac{d}{dx})\right] F(x)$$

converges on some interval, then $F(x)$ can be extended to an entire function for any α and in this case, the entire function obeys

$$|\mathcal{F}(x)| \leq \epsilon e^{|x|-(\alpha-1)\,\mathrm{Re}\,x} + C(\epsilon)\epsilon e^{-(\alpha-1)\,\mathrm{Re}\,x}.$$

Therefore, the bound changes, unless x lies on the imaginary axis [21]. To our knowledge that is the only result in the literature that "quantify" the role of the parameter α. We shall return to this point in what follows. We also recall that differently from all the other works discussed here, Fischer and Kolář make explicit reference to the fact that the tangent series represents the difference between $\mathrm{Re}\, F_+/s$ and a constant associated with the subtraction. Moreover, they also discuss the contribution from the integral with zero and s_0 as lower and upper limits, respectively (the high-energy approximation). They have also shown that for entire functions associated with fits, this contribution is divergent unless $\mathrm{Im}\, F(0) = 0$ (a result also presented in [18]).

It should be also mentioned that Block and Cahn obtained a result for the DDR that does not depend on α, but under the assumption that the differential operator $Z = \mathrm{d}/\,\mathrm{d}\ln s$ follows the condition $|Z| < 1$ [2].

We conclude this discussion with the following comments. Concerning the range and validity of the parameter α, we have shown that the different statements, briefly quoted in Section 2, are not controversial since they are consequences of the different assumptions or classes of functions considered. As shown by Fischer and Kolář, for functions that are entire in $\ln s$, the DDR are continuation of the IDR for all values of the parameter α. However, based on the above bounds, established for $\alpha = 1$ and $\alpha \neq 1$, we can not devise any practical advantage in using $\alpha \neq 1$, as stated before by Fischer and Kolář. In this sense, we understand that the introduction of the parameter α by BKS is only an unnecessary complication.

With the exception of the analysis by Fischer and Kolář, neither of the works discussed in this section make any reference to the subtraction constant nor treat explicitly the contribution from the high-energy approximation represented by the assumption $s_0 \to 0$. In the following section we investigate the practical role of the subtraction constant and the influence of the lower limit of the IDR for a class of functions that are entire in $\ln s$.

4 Tests with Pomeron-Reggeon parametrizations

In order to quantitatively investigate the practical applicability of the DDR in substitution to IDR, we consider, as a framework, some standard parametrizations for the total cross section and present a detailed study on simultaneous fits to $\sigma_{\text{tot}}(s)$ and $\rho(s)$. We start this Section with some comments about our choices concerning the parametrizations, ensemble of experimental data, energy cutoffs and an outline on the strategies to be used in this study. We then present the fit results, followed by a discussion on the role of the subtraction constant and of the high-energy approximation associated with the lower integral limit.

4.1 Basic comments

Presently, phenomenological analyses on the forward quantities $\sigma_{\text{tot}}(s)$ and $\rho(s)$, at high energies, are based on two kinds of dependences on the energy, namely power and power logarithmic functions and, therefore, entire functions in $\ln s$. In the context of the Regge phenomenology they are associated with simple-pole Pomeron and secondary reggeons ($s^{\pm\gamma}$, $0 < \gamma < 1$), double-pole ($\ln s$) and triple-pole ($\ln^2 s$) contributions [16] . In order to treat in detail the applicability of the DDR and IDR by means of a complete example, we shall consider here, as a framework, only parametrizations based on simple-pole Pomeron and secondary reggeons, leaving the other cases for a forthcoming work.

Although that Pomeron contribution, represented by the s^{ϵ} dependence, with $\epsilon \approx 0.081$ (Section 4.2), eventually violates the Froissart-Martin bound, we shall focus here only on its mathematical character, as representing a class of functions entire in $\ln s$ and, moreover, that can provide a good description of the experimental data at the energies presently available. It should be also noted that other forms of DDR do not require the above bound, as demonstrated in [21].

Since all the Regge phenomenology is intended for the region of high energies [35] and, as mentioned previously, for particle-particle and particle-antiparticle (crossing) the pp and $\bar{p}p$ scattering correspond to the highest energy values with available data, we shall concentrate here only on these two processes. We shall return to this point in Section 4.4. We make use of the data sets on σ_{tot} and ρ analyzed and compiled by the Particle Data Group [36]. The statistic and systematic errors have been added in quadrature.

Since we are interested in the practical role of the subtraction constant as a free fit parameter, it is necessary to test both the different cutoffs in the

energy and the effect of the number of free fit parameters involved. To treat the former case, we shall consider in all the analysis two lower energy cuts: $\sqrt{s}_{\rm min} = 5$ GeV and $\sqrt{s}_{\rm min} = 10$ GeV. In the later case, in order to get quantitative information on the sensitivity of the subtraction constant as a free fit parameter, we first consider an "economical" parametrization represented by two reggeon exchanges (the Donnachie-Landshoff model) and after, an extended model with three reggeon exchanges. In the Regge context that means to consider *degenerate* and *non-degenerate* higher *meson trajectories*, respectively.

Our aim in this section is to use these parametrizations in order to discuss the applicability of both the IDR, either with $s_0 = 2m^2$ or $s_0 = 0$, and the DDR. Specifically, we want to establish how the results with the DDR deviate from those with the IDR and in which circumstances both approaches lead to the same results. Summarizing, our strategy shall be to consider simultaneous fits to $\sigma_{\rm tot}$ and ρ data from pp and $\bar{p}p$ scattering, with the following variants: (i) parametrizations with degenerate and non-degenerate trajectories; (ii) lower energy cuts at $\sqrt{s}_{\rm min} = 5$ and $\sqrt{s}_{\rm min} = 10$ GeV; (iii) subtraction constant: $K = 0$ and K as a free fit parameter; (iv) DDR and IDR with both $s_0 = 2m^2$ and with $s_0 = 0$.

In what follows, we consider the approximation $s = 4(k^2 + m^2) \sim 4k^2$, so that the optical theorem, Eq. (2) reads

$$\sigma_{\rm tot}(s) = \frac{{\rm Im}\, F(s, t = 0)}{s}. \tag{14}$$

4.2 Degenerate meson trajectories

The forward effective Regge amplitude introduced by *Donnachie and Landshoff* (DL) has two contributions, one from a single Pomeron and the other from secondary Reggeons exchanges [37]. The parametrization assumes degeneracies between the secondary reggeons, imposing a common intercept for the $C = +1$ (a_2, f_2) and the $C = -1$ (ω, ρ) and is given by

$$\sigma_{\rm tot}^{pp}(s) = X s^{\epsilon} + Y s^{-\eta}, \tag{15}$$

$$\sigma_{\rm tot}^{p\bar{p}}(s) = X s^{\epsilon} + Z s^{-\eta}. \tag{16}$$

Here, $\epsilon = \alpha_{\mathbb{P}}(0) - 1$ and $\eta = 1 - \alpha_{\mathbb{R}}(0)$, where $\alpha_{\mathbb{P}}(0)$ and $\alpha_{\mathbb{R}}(0)$ are the Pomeron and Reggeon intercepts, respectively. From analysis of pp and $\bar{p}p$ in the interval $10\ \text{GeV} \leq \sqrt{s} \leq 546\ \text{GeV}$, DL obtained the following values for the free parameters: $X = 21.7$ mb, $Y = 56.08$ mb, $Z = 98.39$ mb, $\epsilon = 0.0808$, $\eta = 0.4525$.

In what follows, we first present the analytical connections between $\sigma_{\rm tot}$ and ρ with the DL parametrization, using both the IDR (with $s_0 = 2m^2$ or $s_0 = 0$) and the DDR and then, the results of global (simultaneous) fits to $\sigma_{\rm tot}$ and ρ, with $K = 0$ or K as a free fit parameter and energy cutoffs at 5 and 10 GeV.

4.2.1 Analytical results

The crossing even and odd amplitudes in Eq. (1) are obtained by using the parametrizations (15) and (16) and the optical theorem as given by Eq. (14). In this case, $\mathrm{Im}\, F_{+/-}(s) \propto s^{\gamma}$, with $-1 < \gamma < 1$ and the integrals in (4) and (5) may be analytically evaluated. For $s_0 = 0$ we obtain,

$$P \int_0^{+\infty} \mathrm{d}s' \frac{s'^{\gamma}}{(s'^2 - s^2)} = \frac{\pi}{2} s^{\gamma-1} \tan\frac{\pi\gamma}{2}.$$

For $s_0 = 2m^2$ fixed, the change of variables $s' = y + s_0$ and integration in y, lead to the result

$$P \int_{s_0}^{+\infty} \frac{s'^{\gamma}}{s'^2 - s^2} \mathrm{d}s' = \frac{\pi}{2} s^{\gamma-1} \tan\frac{\pi\gamma}{2} + L(s),$$

where, from the above formulas, $L(s)$ represents the contribution below the lower integration limit,

$$L(s) = -P \int_0^{s_0} \mathrm{d}s' \frac{s'^{\gamma}}{(s'^2 - s^2)} = \frac{\pi}{2} s^{\gamma-1} \tan\frac{\pi\gamma}{2},$$

and can be expressed either interms of hypergeometric functions or the corresponding series expansion in inverse powers of s [38]

$$\begin{aligned} L(s) &= \frac{{s_0}^{\gamma}}{2\gamma s} \left[{}_2\mathrm{F}_1(1, \gamma; 1+\gamma; s_0/s) - {}_2\mathrm{F}_1(1, \gamma; 1+\gamma; -s_0/s) \right] \\ &= \frac{{s_0}^{\gamma}}{s} \sum_{j=0}^{\infty} \frac{1}{2j+1+\gamma} \left(\frac{s_0}{s}\right)^{2j+1}. \end{aligned}$$

With these expressions, and the DL parametrization, the connections between $\sigma_{\rm tot}(s)$ and $\rho(s)$ may be analytically determined. For $s_0 = 0$ we obtain

$$\rho(s)\sigma_{\rm tot}(s) = \frac{K}{s} \pm \frac{(Y - Z)}{2} s^{-\eta} \cot\left(\eta\frac{\pi}{2}\right) + X s^{\epsilon} \tan\left(\epsilon\frac{\pi}{2}\right)$$

$$- \frac{(Y+Z)}{2} s^{-\eta} \tan\left(\eta \frac{\pi}{2}\right), \tag{17}$$

and for fixed $s_0 = 2m^2$,

$$\begin{aligned}
\rho(s)\sigma_{\text{tot}}(s) = & \frac{K}{s} \pm \frac{(Y-Z)}{2} s^{-\eta} \cot\left(\eta \frac{\pi}{2}\right) + X s^{\epsilon} \tan\left(\epsilon \frac{\pi}{2}\right) \\
& - \frac{(Y+Z)}{2} s^{-\eta} \tan\left(\eta \frac{\pi}{2}\right) + \frac{1}{\pi} \sum_{j=0}^{\infty} \Bigg(\pm \frac{(Y-Z) {s_0}^{1-\eta}}{s(2j+2-\eta)} \\
& + \frac{2X {s_0}^{\epsilon}}{2j+1+\epsilon} + \frac{(Y+Z) {s_0}^{-\eta}}{2j+1-\eta} \Bigg) \left(\frac{s_0}{s}\right)^{2j+1},
\end{aligned} \tag{18}$$

where the signs $\pm$ apply for pp $(+)$ and $\bar{p}p$ $(-)$ scattering and, as before, the power series is associated with the contribution below the lower integration limit.

The results with the DDR, as given by Eqs. (10) and (11) (or (12) and (13)), are exactly the same as that obtained by means of the IDR with $s_0 = 0$, namely, Eq. (17). This is in agreement with our conclusion at the end of the Section 3.1: the only formal approximation involved in going from IDR to DDR is to take $s_0 = 2m^2 \to 0$.

Once we have established the analytical equivalence of the real parts obtained with both DDR and IDR for $s_0 = 0$, in what folllows we shall consider only the results provided by DDR, Eq. (17), and by IDR for $s_0 = 2m^2$, Eq. (18).

4.2.2 Global Fits

We have developed global (simultaneous) fits to σ_{tot} and ρ with the DL parametrization, using either the IDR with $s_0 = 2m^2$, or the DDR, considering either $K = 0$ or K as a free fit parameter and for energy cuts at 5 and 10 GeV. The fits have been performed with the program CERN Minuit and the errors in the fit parameters correspond to an increase of the χ^2 by one unit. The numerical results and statistical information from all the four cases analyzed with energy cut at 5 GeV are displayed in Table 1, and the curves, together with the experimental data, are shown in Figs. 1 and 2, for $K = 0$ and K as a free fit parameter, respectively. The corresponding results for the energy cut at 10 GeV are presented in Table 2 and Figs. 3 and 4. We shall discuss these results in Section 4.4 together with those obtained by means of an extended parametrization, characterized by the non-degenerate trajectories.

Table 1
Simultaneous fits to σ_{tot} and ρ through the DL parametrization, $\sqrt{s}_{\text{min}} = 5$ GeV (238 data points), either with K as a free parameter or $K = 0$ and using IDR with lower limit $s_0 = 2m^2$ and DDR.

	IDR with $s_0 = 2m^2$		DDR	
	K free	$K = 0$	K free	$K = 0$
X (mb)	23.08 ± 0.28	23.14 ± 0.28	23.08 ± 0.28	23.17 ± 0.28
Y (mb)	54.65 ± 1.3	54.4 ± 1.3	54.66 ± 1.3	48.8 ± 1.0
Z (mb)	108.8 ± 3.5	108.5 ± 3.5	108.8 ± 3.6	95.0 ± 2.4
ϵ	0.0747 ± 0.0014	0.0743 ± 0.0014	0.0747 ± 0.0014	0.0738 ±0.0014
η	0.494 ± 0.010	0.4946 ± 0.0099	0.494 ± 0.010	0.4673 ± 0.0079
K	17 ± 13	0	179 ± 15	0
χ^2/F	1.39	1.39	1.39	2.01

Table 2
Simultaneous fits to σ_{tot} and ρ through the DL parametrization, $\sqrt{s}_{\text{min}} = 10$ GeV (154 data points), either with K as a free parameter or $K = 0$ and using IDR with lower limit $s_0 = 2m^2$ and DDR.

	IDR with $s_0 = 2m^2$		DDR	
	K free	$K = 0$	K free	$K = 0$
X (mb)	21.62 ± 0.38	21.78 ± 0.37	21.62 ± 0.38	21.83 ± 0.38
Y (mb)	65.7 ± 4.5	59.5 ± 3.2	65.7 ± 4.5	50.6 ± 2.2
Z (mb)	113.4 ± 9.4	102.5 ± 7.0	113.4 ± 9.4	85.7 ± 4.4
ϵ	0.0816 ± 0.0018	0.0806 ± 0.0018	0.0816 ± 0.0018	0.0800 ± 0.0018
η	0.482 ± 0.019	0.467 ± 0.017	0.482 ± 0.019	0.435 ± 0.014
K	116 ± 36	0	287 ± 44	0
χ^2/F	1.17	1.24	1.17	1.50

4.3 Non-degenerate meson trajectories

Analyses, treating global fits to σ_{tot} and ρ, have indicated that the best results are obtained with non-degenerate meson trajectories [39]. In this case the forward scattering amplitude is decomposed into three reggeon exchanges, $F(s) = F_{\mathbb{P}}(s) + F_{a_2/f_2}(s) + \tau F_{\omega/\rho}(s)$, where the first term represents the exchange of a single Pomeron, the other two the secondary Reggeons and $\tau = +1$ (-1) for pp ($\bar{p}p$) amplitudes. Using the notation $\alpha_{\mathbb{P}}(0) = 1+\epsilon$, $\alpha_+(0) = 1-\eta_+$ and $\alpha_-(0) = 1 - \eta_-$ for the intercepts of the Pomeron and the $C = +1$ and $C = -1$ trajectories, respectively, the total cross sections, Eq. (14), for pp and

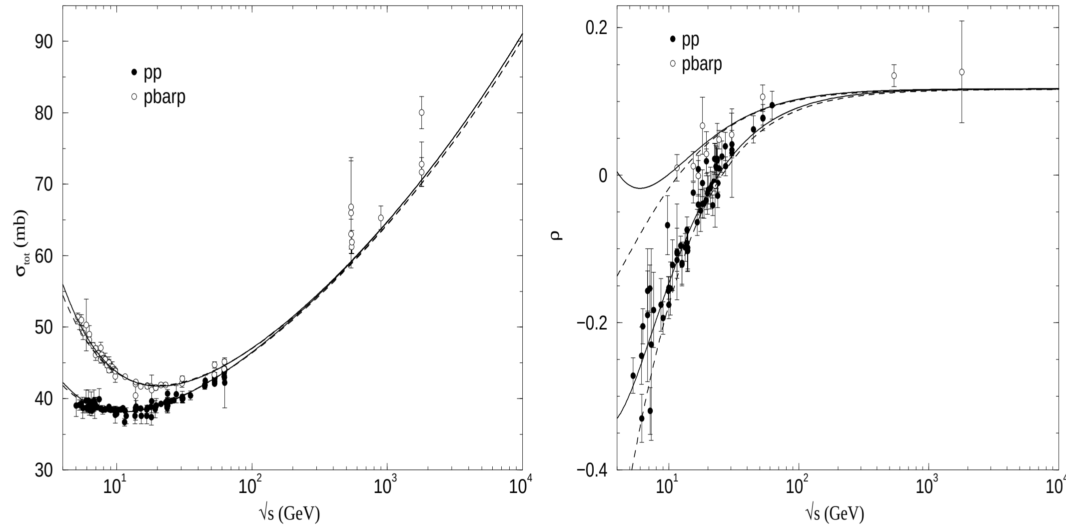

Fig. 1. Simultaneous fit to $\sigma_{\rm tot}$ and ρ through the DL parametrization, $\sqrt{s}_{\rm min} = 5$ GeV, assuming $K = 0$ and using either the IDR with $s_0 = 2m^2$ (solid) or the DDR (dashed).

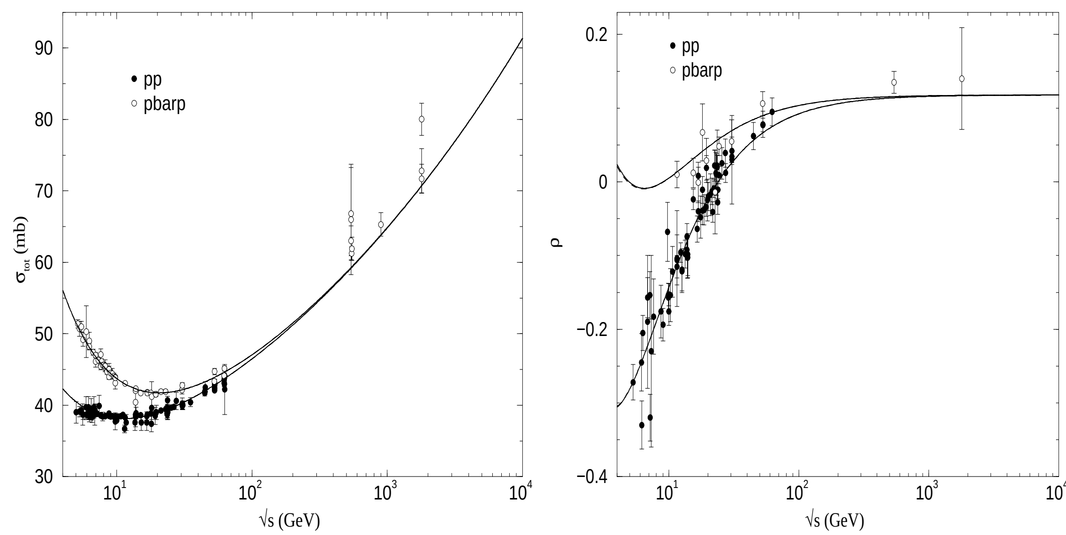

Fig. 2. Simultaneous fit to $\sigma_{\rm tot}$ and ρ through the DL parametrization, $\sqrt{s}_{\rm min} = 5$ GeV, with K as free parameter and using either the IDR with $s_0 = 2m^2$ (solid) or the DDR (dashed): both curves coincide.

$\bar{p}p$ interactions are written as

$$\sigma_{\rm tot}(s) = X s^{\epsilon} + Y_+ \, s^{-\eta_+} + \tau Y_- \, s^{-\eta_-}. \tag{19}$$

Since the formal structure of the parametrization is the same as in the case of degenerate trajectories, with the addition of one more power term, we shall not display the analytical formulas, but only the numerical and fit results, together with the experimental data. The numerical results and statistical information

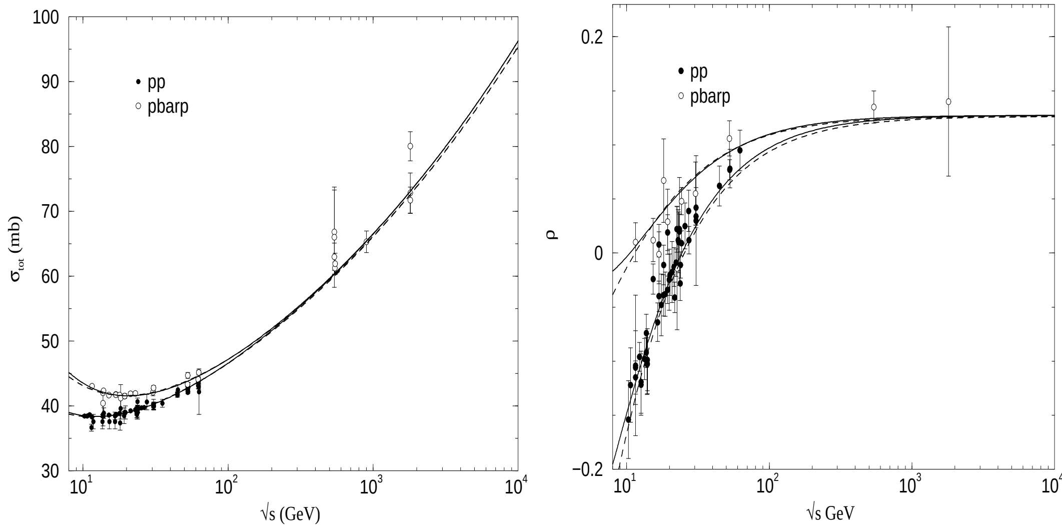

Fig. 3. Simultaneous fit to $\sigma_{\rm tot}$ and ρ through the DL parametrization, $\sqrt{s}_{\rm min} = 10$ GeV, assuming $K = 0$ and using either the IDR with $s_0 = 2m^2$ (solid) or the DDR (dashed).

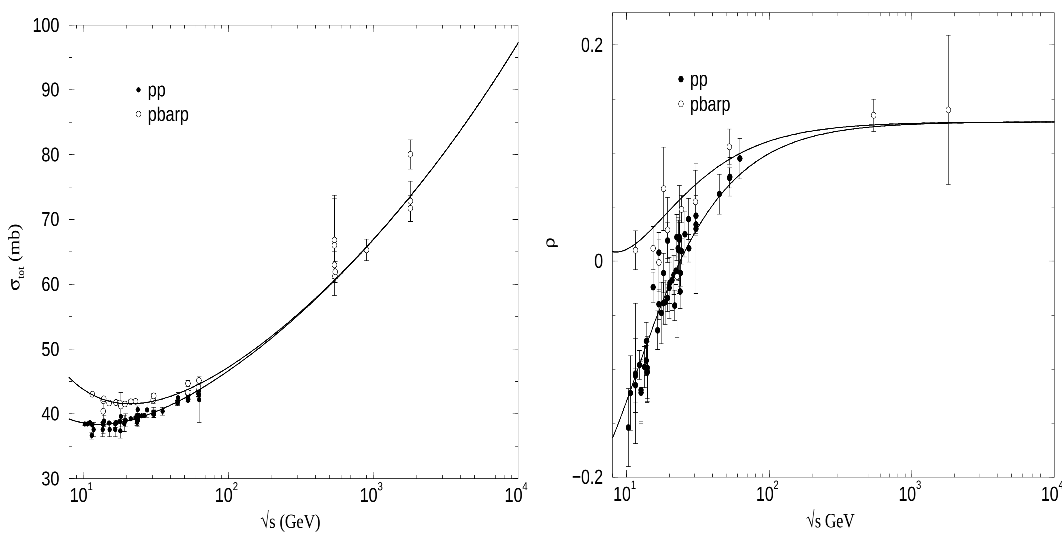

Fig. 4. Simultaneous fit to $\sigma_{\rm tot}$ and ρ through the DL parametrization, $\sqrt{s}_{\rm min} = 10$ GeV, with K as free parameter and using either the IDR with $s_0 = 2m^2$ (solid) or the DDR (dashed): both curves coincide.

from all the four cases analyzed with $\sqrt{s}_{\rm min} = 5$ GeV are displayed in Table 3 and the curves, together with the experimental data, are shown in Figs. 5 and 6, for $K = 0$ or K as a free fit parameter, respectively. The corresponding results for $\sqrt{s}_{\rm min} = 10$ GeV are presented in Table 4 and Figs. 7 and 8.

Table 3
Simultaneous fits to $\sigma_{\rm tot}$ and ρ through the extended parametrization, $\sqrt{s}_{\rm min} = 5$ GeV (238 data points), either with K as a free parameter or $K = 0$ and using IDR with lower limit $s_0 = 2m^2$ and DDR.

	IDR with $s_0 = 2m^2$		DDR	
	K free	$K = 0$	K free	$K = 0$
X (mb)	19.61 ± 0.53	20.12 ± 0.51	19.61 ± 0.53	21.44 ± 0.44
Y_+ (mb)	66.4 ± 2.0	68.9 ± 2.1	66.4 ± 2.0	78.0 ± 2.0
Y_- (mb)	-33.8 ± 1.9	-32.4 ± 1.8	-33.8 ± 1.9	-38.8 ± 2.6
ϵ	0.0895 ± 0.0025	0.0874 ± 0.0024	0.0895 ± 0.0025	0.0814 ± 0.0021
η_+	0.382 ± 0.015	0.399 ± 0.014	0.382 ± 0.015	0.450 ± 0.012
η_-	0.545 ± 0.013	0.533 ± 0.012	0.545 ± 0.013	0.573 ± 0.015
K	-48 ± 15	0	69 ± 18	0
χ^2/F	1.10	1.14	1.10	1.64

Table 4
Simultaneous fits to $\sigma_{\rm tot}$ and ρ through the extended parametrization, $\sqrt{s}_{\rm min} = 10$ GeV (154 data points), either with K as a free parameter or $K = 0$ and using IDR with lower limit $s_0 = 2m^2$ and DDR.

	IDR with $s_0 = 2m^2$		DDR	
	K free	$K = 0$	K free	$K = 0$
X (mb)	19.57 ± 0.79	19.70 ± 0.64	19.58 ± 0.78	21.088 ± 0.54
Y_+ (mb)	66.0 ± 6.7	67.4 ± 4.8	66.0 ± 6.6	86.850 ± 4.4
Y_- (mb)	-29.2 ± 4.0	-29.1 ± 3.9	-29.2 ± 4.0	-27.477 ± 6.0
ϵ	0.0897 ± 0.0033	0.0892 ± 0.0028	0.0897 ± 0.0033	0.0836 ± 0.0024
η_+	0.380 ± 0.033	0.386 ± 0.024	0.380 ± 0.033	0.46256 ± 0.019
η_-	0.520 ± 0.025	0.519 ± 0.024	0.520 ± 0.024	0.50596 ± 0.040
K	-14 ± 48	0	104 ± 58	0
χ^2/F	1.10	1.09	1.10	1.55

4.4 Discussion

Making use of power law parametrizations for the total cross section, we have presented the results of several simultaneous fits to $\sigma_{\rm tot}(s)$ and $\rho(s)$ from pp and $\bar{p}p$ scattering. Once established that the analytical results through IDR with $s_0 = 0$ and DDR are the same (Section 4.2.1), we have performed 16 fits to the experimental data with the following variants: (1) degenerate and

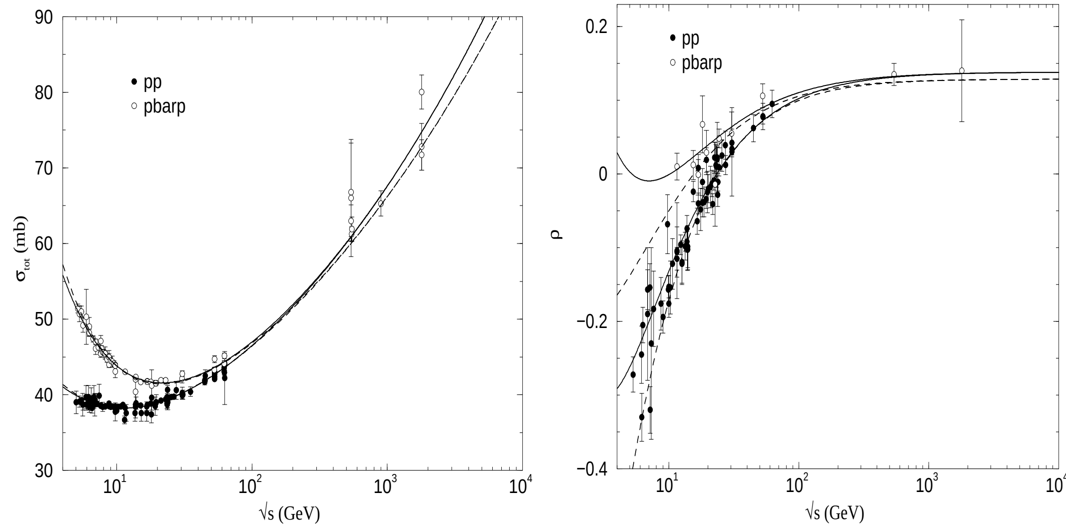

Fig. 5. Simultaneous fit to σ_{tot} and ρ through the extended parametrization, $\sqrt{s}_{\text{min}} = 5$ GeV, assuming $K = 0$ and using either the IDR with $s_0 = 2m^2$ (solid) or the DDR (dashed).

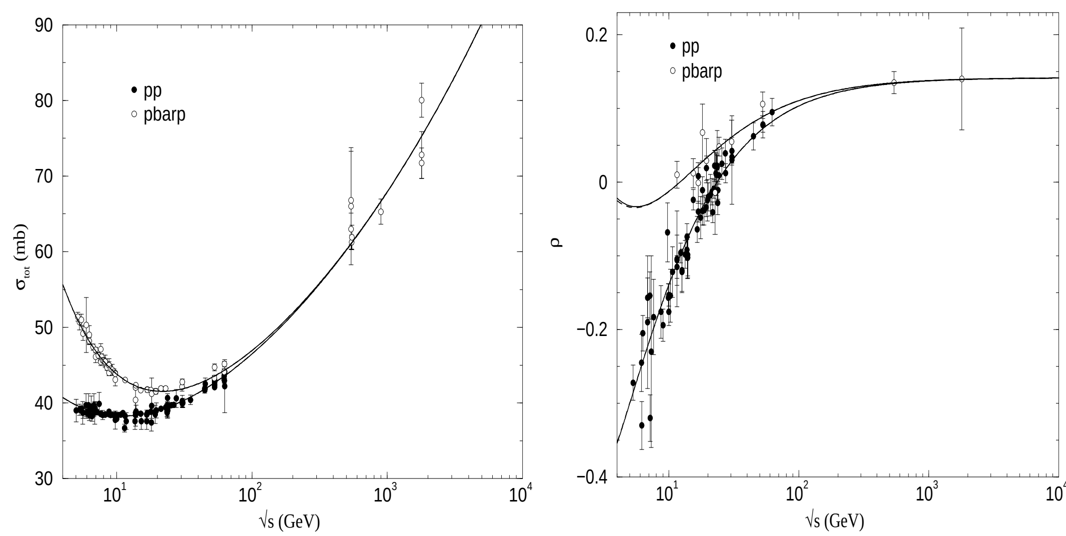

Fig. 6. Simultaneous fit to σ_{tot} and ρ through the extended parametrization, $\sqrt{s}_{\text{min}} = 5$ GeV, with K as free parameter and using either the IDR with $s_0 = 2m^2$ (solid) or the DDR (dashed): both curves coincide.

non-degenerate trajectories; (2) $\sqrt{s}_{\text{min}} = 5$ GeV and $\sqrt{s}_{\text{min}} = 10$ GeV; (3) IDR with $s_0 = 2m^2$ and DDR; (4) $K = 0$ and K as a free fit parameter. The results are displayed in Tables 1 to 4 and Figures 1 to 8. In what follows we focus the discussion on the effect of the high-energy approximation ($s_0 \to 0$ in the IDR leading to the DDR) and on the role of the subtraction constant.

For $K = 0$, the only difference between the IDR and the DDR approaches is associated with the high-energy approximation, analytically represented by integrations of Eqs. (4) and (5) from $s' = 0$ to $s' = s_0 = 2m^2 \approx 1.8$ GeV. In

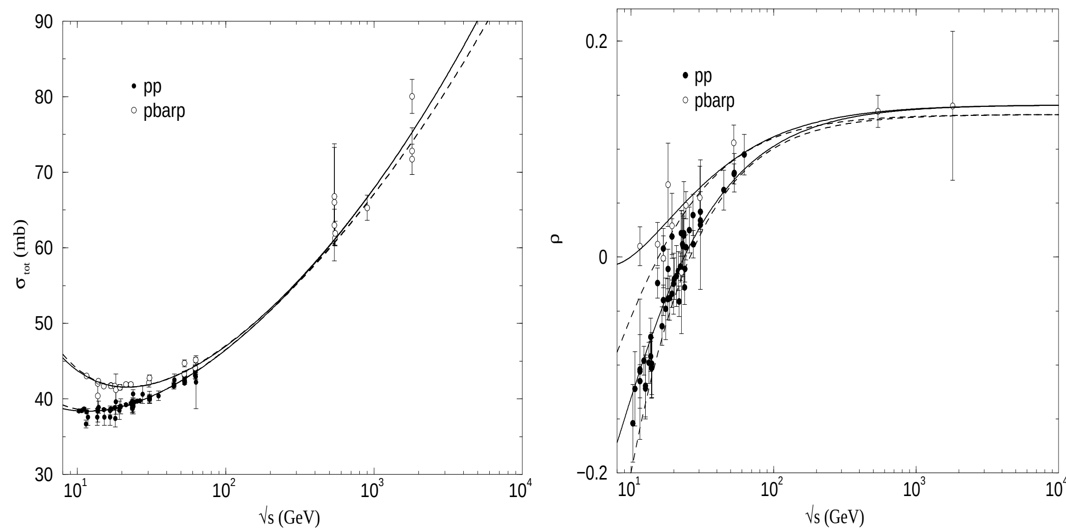

Fig. 7. Simultaneous fit to σ_{tot} and ρ through the extended parametrization, $\sqrt{s}_{\mathrm{min}} = 10$ GeV, assuming $K = 0$ and using either the IDR with $s_0 = 2m^2$ (solid) or the DDR (dashed).

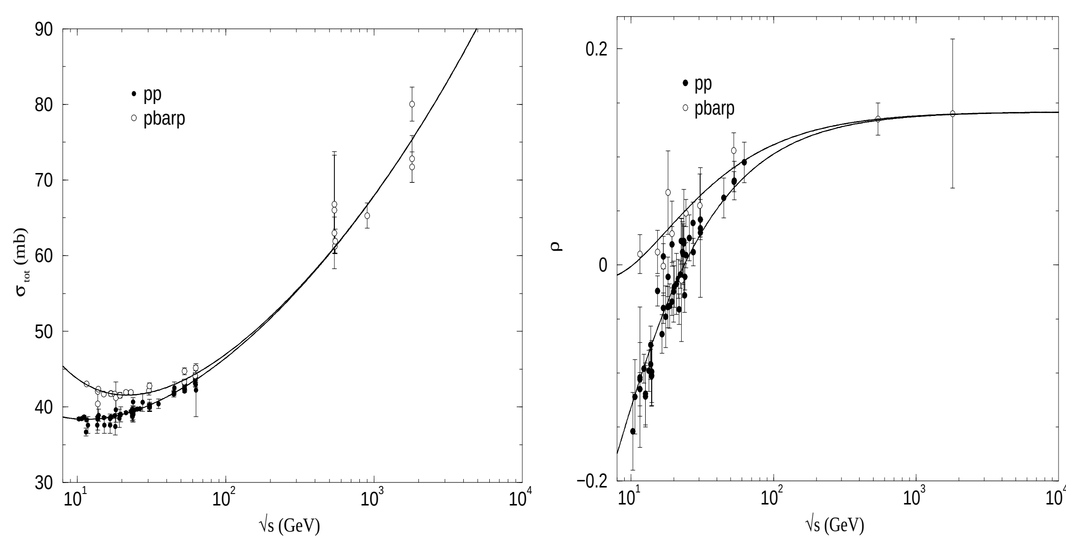

Fig. 8. Simultaneous fit to σ_{tot} and ρ through the extended parametrization, $\sqrt{s}_{\mathrm{min}} = 10$ GeV, with K as free parameter and using either the IDR with $s_0 = 2m^2$ (solid) or the DDR (dashed): both curves coincide.

principle, it might be expected that this contribution could affect the fit results only in the low-energy region, roughly below $\sqrt{s} \approx 50$ GeV. However, the fit procedure is characterized by strong correlations among the free parameters and, therefore, the contributions from any region may be “communicated” to other regions and that is exactly what our results show. In fact, for $K = 0$, independently of the parametrization and the lower energy cut used, the leading contribution at high energies (represented by the value of the Pomeron intercept, ϵ) is different when using IDR or DDR. The above approximation (DDR) results in a value for the intercept that is smaller than that obtained

with the IDR. Specifically, in the case of the DL parametrization (Tables 1 and 2), the reduction reads 0.67 % for $\sqrt{s}_{\rm min} = 5$ GeV and 0.7 % for $\sqrt{s}_{\rm min} = 10$ GeV (see $\sigma_{\rm tot}(s)$ in Figs 1 and 3). The effect is more significant in the case of the extended parametrization (Tables 3 and 4), with a reduction of 6.9 % for $\sqrt{s}_{\rm min} = 5$ GeV and 6.3 % for $\sqrt{s}_{\rm min} = 10$ GeV (see $\sigma_{\rm tot}(s)$ in Figs. 5 and 7). That is an important and novel result, which shows that, in practice, for the simple-pole Pomeron and secondary reggeons, the high-energy approximation enclosed in the DDR affects the fits results even at the asymptotic energies. As mentioned, that is a consequence of the fit procedure and the correlations among the free parameters.

Now, the practical role of the subtraction constant may be investigated by comparing the results obtained with $K = 0$ and those with K as a free fit parameter. The striking result here is the fact that, independently of the parametrization and the energy cut used, the numerical results obtained for the fit parameters with the IDR and with the DDR are the same up to 3 significant figures, including the χ^2/F. For example, from Tables 1 and 2, for the case of the DL parametrization, we have obtained in both cases (IDR with $s_0 = 2m^2$ and DDR) the same values: $\epsilon = 0.0747 \pm 0.0014$, $\chi^2/F = 1.39$ for $\sqrt{s}_{\rm min} = 5$ GeV and $\epsilon = 0.0816 \pm 0.0018$, $\chi^2/F = 1.17$ for $\sqrt{s}_{\rm min} = 10$ GeV (see Figs 2 and 4, respectively). For the cases of the extended parametrization, Tables 3 and 4, we have obtained $\epsilon = 0.0895 \pm 0.0025$, $\chi^2/F = 1.10$ for $\sqrt{s}_{\rm min} = 5$ GeV and $\epsilon = 0.0897 \pm 0.0033$, $\chi^2/F = 1.10$ for $\sqrt{s}_{\rm min} = 10$ GeV (see Figs 6 and 8, respectively).

All these numerical results can be understood in an analytical context if we investigate the correlation between the subtraction constant and the contribution below the lower integration limit (referred to in Sec. 4.2.1). To this end, let us return to the product of $\rho(s)$ by $\sigma_{tot}(s)$ in the case of IDR with $s_0 = 2m^2$, as given by Eq. (18), or the corresponding formula with non-degenerate trajectories. In both cases the terms with entire inverse powers of s can be expressed by

$$[K + \Delta]\frac{1}{s} + \mathcal{O}(1/s^2),$$

where K is the subtraction constant and Δ comes from the series expansion, which is associated with the integration from $s' = 0$ to $s' = 2m^2$ (Sec. 4.2.1). Therefore, at large enough energies, this contribution below the lower integration limit is expected to be absorbed in the subtraction constant. In fact, with degenerate (de) and non-degenerate (non-de) trajectories, these contributions read

$$\Delta_{\rm de} = \frac{1}{\pi}\left(\frac{2X{s_0}^{1+\epsilon}}{1+\epsilon} + \frac{(Y+Z){s_0}^{1-\eta}}{1-\eta}\right),$$

$$\Delta_{\rm non-de} = \frac{1}{\pi}\left(\frac{2X{s_0}^{1+\epsilon}}{1+\epsilon} + \frac{2Y_+{s_0}^{1-\eta_+}}{1-\eta_+}\right),$$

and may be estimated with the central values of the fit parameters from Tables 1-4:

$\Delta_{\rm de} \approx 162 \ \ (\sqrt{s}_{\rm min} = 5 \text{ GeV}), \qquad \Delta_{\rm de} \approx 170 \ \ (\sqrt{s}_{\rm min} = 10 \text{ GeV}),$

$\Delta_{\rm non-de} \approx 118 \ \ (\sqrt{s}_{\rm min} = 5 \text{ GeV}), \qquad \Delta_{\rm non-de} \approx 117 \ \ (\sqrt{s}_{\rm min} = 10 \text{ GeV}),$

From those Tables, we see that these values are in complete agreement with the differences between the fitted values of the subtraction constants determined with DDR and IDR for $s_0 = 2m^2$. Therefore, we conclude that in the case of DDR we have an "effective" subtraction constant which can compensate (analytically and numerically) the effect of the high-energy approximation.

That is another important and novel result, showing explictly the practical role of this parameter: for simple-pole pomeron and secondary reggeons once the subtraction constant is considered as a free fit parameter, the DDR are completely equivalent to the IDR with fixed (non-zero) lower limit. We also note that our qualitative and quantitative conclusions about the effects of the high-energy approximation and the subtraction constant are the same with both the degenerate and non-degenerate trajectories and both energy cuts, 5 and 10 GeV.

To conclude this Section, let us call attention to some critical aspects related with the use of the DL and the extended parametrizations. From a statistical point of view, the extended parametrization leads to a better description of the experimental data (Tables 1-4). Also, the ρ data at 546 GeV and 1.8 TeV are well described with the extended parametrization, but not in the DL case. The essential difference between these two parametrizations concerns the additional secondary reggeon, represented by the dependence $s^{-\gamma}$, $0 < \gamma < 1$, which goes to zero as the energy increases. We conclude that the splitting of the trajectories allows the free parameters from the secondary reggeons to fit the data at low energies (between 5 GeV and, let us say, 100 GeV), giving more freedom for the parameters associated with the Pomeron to fit the data at the highest energies. That seems to be well known and also well accepted in the literature [39]. However, we can not forget that the Regge phenomenology is intended only for asymptotic energies. Specifically, the basic contribution comes from the asymptotic form of the Legendre Polynomial, namely $P_l(x) \to x^l \quad \text{as} \quad x \to \infty$, where x is translated to the energy s through the crossing symmetry and l to the trajectory of the Pomeron or Reggeon [35]. On the other hand, from the above discussion, the secondary reggeons act in a region where the total cross section reaches its minimum (followed by an increase) and the ρ parameter cross the zero (becoming positive). Even if we, optimistically, concentrate in the above region, $\sqrt{s} = 5$ to 100 GeV, that

region, certainly, has nothing to do with an asymptotic concept. Based on these facts, we understand that, although the secondary reggeons can be used as suitable parametrizations on statistical or strictly fit grounds, one must be careful in attempting to extract well founded physical results from these asymptotic forms used at so low values of the energy. On the other hand, that is not the case for the subtraction constant as a free fit parameter, since it has well defined mathematical bases, which are related with polynomial bounds.

5 Conclusions and final remarks

In this work we have presented a review on the different results and statements in the literature related with the replacement of IDR by DDR, and a discussion connecting these different aspects with the corresponding assumptions and classes of functions considered in each case.

By means of a formal and analytical approach, we have demonstrated that the subtraction constant is preserved when the IDR are replaced by the DDR and that, for the class of functions entire in $\ln s$, the DDR do not depend on any additional free parameter (except for the subtraction constant). We have stressed that the only approximation involved in this replacement concerns the lower limit s_0 in the IDR: the high-energy condition is reached by assuming that $s_0 = 2m^2 \to 0$.

We have investigated the practical applicability of the DDR and IDR in the context of the Pomeron-reggeon parametrizations, with both degenerate and non-degenerate higher meson trajectories. By means of global fits to $\sigma_{\mathrm{tot}}(s)$ and $\rho(s)$ data from pp and $\bar{p}p$ scattering, we have tested all the important variants that could affect the fit results, namely the number of secondary reggeons, energy cutoff, effects of the high-energy approximation and the subtraction constant and the derivative approach with DDR and IDR with fixed s_0. Our results lead to the conclusion that the high-energy approximation and the subtraction constant affect the fit results at both low and high energies. This effect is a consequence of the fit procedure, associated with the strong correlation among the free parameters.

A striking novel result concerns the practical role of the subtraction constant. We have shown that, with the Pomeron-reggeon parametrization, once the subtraction constant is used as a free fit parameter, the results obtained with the DDR and with the IDR (with finite lower limit, $s_0 = 2m^2$) are the same up to 3 significant figures in the fit parameters and χ^2/F. Analytically this effect is due to the absorption of the high-energy approximation in an "effective" subtraction constant. This conclusion, as we have shown, is independent of the number of secondary reggeons (DL or extended parametrization) or the

energy cutoff ($\sqrt{s} = 5$ or 10 GeV).

We have called the attention to the fact that the subtraction constant has well founded mathematical bases, since it is a consequence of the polynomial bounds in the scattering amplitude. On the other hand, the use of the asymptotic Regge forms for detailed fits at finite energies (minimum on $\sigma_{\rm tot}(s)$) is not formally justified. We understand that our results suggest that, presently, it is very important to look for usefull and well founded theoretical results at finite energies and that one must be careful in attempting to predict asymptotic behaviors based on asymptotic formalisms applied at finite energies.

In order to treat a complete example, with the 16 variants referred to in the text, we have considered here only a simple-pole representation for the Pomeron. Although belonging to the class of functions entire in $\ln s$, this choice is outside the more general class of functions considered by Fischer and P. Kolář, because it violates the Froissart-Martin bound at the asymptotic energies. We are presently investigating other possibilities, as dipole and tripole contributions. Anyway, there is an interesting aspect concerning these forms, which we shall mention here. The possibility that the odd amplitude had a $\ln s$ contribution (Odderon) has been investigated in Refs. [14,26], by means of the DDR. However, this contribution in the IDR, Eq. (5), with $s_0 = 0$ is divergent. On the other hand, that is not the case if we use the DDR as expressed by Eqs. (9) or (13). Since we have shown that the only approximation in going from IDR to DDR is to take $s_0 \rightarrow 0$ in the integral relation, the above results seems inconsistent. We think that this may be associated with the formal substitution of the tangent by the cotangent operator, as referred to in Sec. 3.1. We are presently investigating this subject.

The DDR introduced by Bronzan, Kane, and Sukhatme [13], and later generalized for an arbitrary number of subtractions by Menon, Motter and Pimentel [23] bring enclosed the parameter α, which we have shown does not need to take part in the calculation. We understand that this dependence is not necessarily wrong if, in practical uses, one takes $\alpha = 1$. Therefore, the presence of this parameter is only an unnecessary complication. Certainly, considering α as a free parameter at finite energies may improve the fit results [13,24], but it seems to us difficult to justify its mathematical and, mainly, its physical meaning.

Note added in proof. After the submission of this work, the effect of the absorption of the high-energy approximation ($s_0 \rightarrow 0$) by the subtraction constant has been confirmed by other authors [40].

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